Lattice theory and basic data flow analysis

Exercise 1.1: 2 Points

Draw a control flow graph for the following fragment of C code.

```c
for (i = 0; i < N; i++) {
    for (j = i; j < N; j++) {
        if (A[i][j] == 0)
            goto exit;
    }
}
exit:
printf("i = %d, j = %d\n", i, j);
```

Edges of the graph should be labeled with statements or branch conditions (\texttt{true(e)} or \texttt{false(e)} for a boolean expression \(e\)). \textbf{Hint:} when in doubt about loops, think how would you rewrite a \texttt{for} loop using \texttt{while}.

Exercise 1.2: 2 Points

1. Prove that if a partially-ordered set has a bottom element, then it is uniquely determined.
2. Prove that every complete lattice has a \(\top\) and \(\bot\) element.

Exercise 1.3: 4 Points

Let \(P\) be a non-empty partially-ordered set. Prove that if \(\bigwedge S\) exists for all subsets \(S \subseteq P\) then \(P\) is a complete lattice. \textbf{Hint:} Show that \(\bigvee S = \bigwedge S^\complement\).

Exercise 1.4: 4 Points

Suppose that \((L, \leq_L)\) is a complete lattice. For each of following definitions of \((P, \leq_P)\) determine whether it is a partial order, lattice, and/or a complete lattice. Does it have a top or bottom element? What is it?

1. \(P = \{X \subseteq \mathbb{N}: |X| \leq 3 \lor X = \mathbb{N}\}\) with ordinary set inclusion \((\leq_P = \subseteq)\).
2. \(P = (L \times L)\) with \((x_1, x_2) \leq_P (y_1, y_2)\) iff \(x_1 \leq_L y_1 \land x_2 \leq_L y_2\).
3. \(P = (\mathbb{N}_+ \times \mathbb{N}_+)\) with \((a, b) \leq_P (a', b')\) iff \(b a' \leq b' a\) where \(\mathbb{N}_+\) is the set of positive natural numbers.
4. \(P = A \cup \{\top, \bot\}\) where \(\top, \bot \in L\) are \(L\)'s top and bottom elements and \(A \subseteq L\) is an arbitrary subset of \(L\). The ordering relation follows the one on \(L\), i.e., \(x \leq_P y\) iff \(x \leq_L y\).