Exercise Sheet 1

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Lattice theory and basic data flow analysis

Exercise 1.1: 2 Points

Draw a control flow graph for the following fragment of C code.

```
for (i = 0; i < N; i++) {
  for (j = i; j < N; j++) {
    if (A[i][j] == 0)
      goto exit;
  }
}
exit:
  printf("i = %d, j = %d\n", i, j);
```

Edges of the graph should be labeled with statements or branch conditions (**true**(e) or **false**(e) for a boolean expression e). Hint: when in doubt about loops, think how would you rewrite a **for** loop using **while**.

Exercise 1.2: 2 Points

1. Prove that if a partially-ordered set has a bottom element, then it is uniquely determined.
2. Prove that every complete lattice has a $\top$ and $\bot$ element.

Exercise 1.3: 4 Points

Let P be a non-empty partially-ordered set. Prove that if $\bigwedge S$ exists for all subsets $S \subseteq P$ then P is a complete lattice. Hint: Show that $\bigvee S = \bigwedge S^u$.

Exercise 1.4: 4 Points

Suppose that $(L, \leq_L)$ is a complete lattice. For each of following definitions of $(P, \leq_P)$ determine whether it is a partial order, lattice, and/or a complete lattice. Does it have a top or bottom element? What is it?

1. $P = \{X \subseteq \mathbb{N} \colon |X| \leq 3 \vee X = \mathbb{N}\}$ with ordinary set inclusion ($\leq_P = \subseteq$).
2. $P = (L \times L)$ with $(x_1, x_2) \leq_P (y_1, y_2)$ iff $x_1 \leq_L y_1 \wedge x_2 \leq_L y_2$.
3. $P = (\mathbb{N}_+ \times \mathbb{N}_+)$ with $(a, b) \leq_P (a', b')$ iff $ba' \leq b'a$ where $\mathbb{N}_+$ is the set of positive natural numbers.
4. $P = A \cup \{\top, \bot\}$ where $\top, \bot \in L$ are L's top and bottom elements and $A \subseteq L$ is an arbitrary subset of L. The ordering relation follows the one on L, i.e., $x \leq_P y$ iff $x \leq_L y$.