Lattices

Slides follow Davey and Priestley: *Introduction to Lattices and Order*

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Partial Orders

Let $P$ be a set. A binary relation $\leq$ on $P$ is a partial order iff it is:

1. **reflexive:** $(\forall x \in P) \ x \leq x$
2. **transitive:** $(\forall x, y, z \in P) \ x \leq y \land y \leq z \implies x \leq z$
3. **antisymmetric:** $(\forall x, y \in P) \ x \leq y \land y \leq x \implies x = y$

An element $\perp$ with $\perp \leq x$ for all $x \in P$ is called **bottom element.** It is unique. Analogously, $\top$ is called **top element,** if $\top \geq x$ for all $x \in P.$
Duality

Let $P$ an ordered set. The dual $P^D$ of $P$ is obtained by defining $x \leq y$ in $P^D$ whenever $y \leq x$ in $P$.

For every statement $\Phi$ about $P$ there is a dual statement $\Phi^D$ about $P^D$. It is obtained from $P$ by exchanging $\leq$ by $\geq$.

If $\Phi$ is true for all ordered sets, $\Phi^D$ is also true for all ordered sets.
Hasse Diagrams

A partial order \((P, \leq)\) is typically visualized by a Hasse diagram:

- Elements of \(P\) are points in the plane
- If \(x \leq z\), then \(z\) is drawn above \(x\).
- If \(x \leq z\), and there is no \(y\) with \(x \leq y \leq z\), then \(x\) and \(z\) are connected by a line

The Hasse diagram of the dual of \(P\) is obtained by “turning” the one of \(P\) by 180°
Upper and Lower Bounds

Let \((P, \leq)\) be a partial ordered set and let \(S \subseteq P\). An element \(x \in P\) is a lower bound of \(S\), if \(x \leq s\) for all \(s \in S\). Let

\[
S^\ell = \{x \in P \mid (\forall s \in S)\, x \leq s\}
\]

be the set of all lower bounds of the set \(S\). Dually:

\[
S^u = \{x \in P \mid (\forall s \in S)\, x \geq s\}
\]

Note: \(\emptyset^u = \emptyset^\ell = P\).

If \(S^\ell\) has a greatest element, this element is called the greatest lower bound and is written \(\text{inf } S\). (Dually for least upper bound and \(\text{sup } S\).) The greatest lower bound only exists, iff there is a \(x \in P\) such that

\[
(\forall y \in P)\, (((\forall s \in S)\, s \geq y) \iff x \geq y)
\]
Complete Partial Orders

A non-empty subset $S \subseteq P$ is directed if for every $x, y \in S$ there is $z \in S$ such that $z \in \{x, y\}^u$.

$P$ is a complete partial order (CPO) if every directed set $M$ has a least upper bound.

We use the notation $\bigsqcup M$ to indicate the least upper bound of a directed set.
Lattices
The order-theoretic definition

Let $P$ be an ordered set.

- If $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for every pair $x, y \in P$ then $P$ is called a lattice.

- If for every $S \subseteq P$, $\sup S$ and $\inf S$ exist, then $P$ is called a complete lattice.
Let $L$ be a lattice and let $a, b \in L$. The following statements are equivalent:

1. $a \leq b$
2. $\inf \{a, b\} = a$
3. $\sup \{a, b\} = b$
Lattices

The algebraic definition

We now view \( L \) as an algebraic structure \((L; \lor, \land)\) with two binary operators

\[
    x \lor y := \sup\{x, y\} \quad x \land y := \inf\{x, y\}
\]

Theorem: \( \lor \) and \( \land \) satisfy for all \( a, b, c \in L \):

\[
\begin{align*}
(L1) & \quad (a \lor b) \lor c = a \lor (b \lor c) \quad \text{associativity} \\
(L1)^D & \quad (a \land b) \land c = a \land (b \land c) \\
(L2) & \quad a \lor b = b \lor a \quad \text{commutativity} \\
(L2)^D & \quad a \land b = b \land a \\
(L3) & \quad a \lor a = a \quad \text{idempotency} \\
(L3)^D & \quad a \land a = a \\
(L4) & \quad a \lor (a \land b) = a \quad \text{absorption} \\
(L4)^D & \quad a \land (a \lor b) = a
\end{align*}
\]

Proof: \((L2)\) is immediate because \( \sup\{x, y\} = \sup\{y, x\} \). \((L3), (L4)\) follow from the connection lemma. \((L1)\) exercise. The dual laws come by duality.
Lattices

The algebraic definition

We now view $L$ as an algebraic structure $(L; \lor, \land)$ with two binary operators

$$x \lor y := \sup\{x, y\} \quad x \land y := \inf\{x, y\}$$

Theorem: $\lor$ and $\land$ satisfy for all $a, b, c \in L$:

(L1) $(a \lor b) \lor c = a \lor (b \lor c)$ \hspace{1cm} associativity

(L1)$^D$ $(a \land b) \land c = a \land (b \land c)$

(L2) $a \lor b = b \lor a$ \hspace{1cm} commutativity

(L2)$^D$ $a \land b = b \land a$

(L3) $a \lor a = a$ \hspace{1cm} idempotency

(L3)$^D$ $a \land a = a$

(L4) $a \lor (a \land b) = a$ \hspace{1cm} absorption

(L4)$^D$ $a \land (a \lor b) = a$

Proof: (L2) is immediate because $\sup\{x, y\} = \sup\{y, x\}$. (L3), (L4) follow from the connection lemma. (L1) exercise. The dual laws come by duality.
Lattices
From the algebraic to the order-theoretic definition

Let \((L; \lor, \land)\) be a set with two operators satisfying \((L1)-(L4)\) and \((L1)^D-(L4)^D\)

Theorem:

1. Define \(a \leq b\) on \(L\) if \(a \lor b = b\). Then, \(\leq\) is a partial order
2. \((L; \leq)\) is a lattice with

\[
\text{sup}\{a, b\} = a \lor b \quad \text{and} \quad \text{inf}\{a, b\} = a \land b
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Lattices
From the algebraic to the order-theoretic definition

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\]

Proof:
1. reflexive by \((L3)\), antisymmetric by \((L2)\), transitive by \((L1)\)
2. First show that \(a \lor b \in \{a, b\}^u\) then show that \(d \in \{a, b\}^u \implies (a \lor b) \leq d\). Easy by applying the \((Li)\) to the suitable premises (Exercise).
Functions on Partial Orders

Let \( P \) be a partial order. A function \( f : P \rightarrow P \) is

- **monotone** if for all \( x, y \in P \):
  \[
  x \leq y \implies f(x) \leq f(y)
  \]

- **continuous** if for each directed subset \( M \subseteq L \):
  \[
  f(\bigcup M) = \bigcup f(M)
  \]

Lemma: Continuous functions are monotone.
Proof: Exercise
Knaster-Tarski Fixpoint Theorem

Let $L$ be a complete lattice and $f : L \to L$ be monotone. Then

$$\bigwedge\{x \in L \mid f(x) \leq x\}$$

is the least fixpoint of $f$. (The dual holds analogously.)
Knaster-Tarski Fixpoint Theorem

Let $L$ be a complete lattice and $f : L \to L$ be monotone. Then

$$\bigwedge \{ x \in L \mid f(x) \leq x \}$$

is the least fixpoint of $f$. (The dual holds analogously.)

Proof: Let $R := \{ x \in L \mid f(x) \leq x \}$ be the set of elements of which $f$ is reductive. Let $x \in R$. Consider $z = \bigwedge R$. $z$ exists, because $L$ is complete. $z \leq x$ because $z$ is a lower bound of $x$. By monotonicity, $f(z) \leq f(x)$. Because $x \in R$, $f(z) \leq x$. Thus, $f(z)$ is also a lower bound of $R$. Thus, $f(z) \leq y$ for all $y \in R$. Because $z$ is the greatest lower bound of $R$, $f(z) \leq z$, thus $z \in R$. By monotonicity, $f(f(z)) \leq f(z)$. Hence, $f(z) \in R$. Because $z$ is a lower bound of $R$, $z \leq f(z)$ and $z = f(z)$. 
Finite Lattices Are Complete

Associativity allows us to write sequences of joins unambiguously without brackets. One can show (by induction) that

$$\bigvee\{a_1, \ldots, a_n\} = a_1 \lor \cdots \lor a_n$$

for \(\{a_1, \ldots, a_n\} \in L, \ n \geq 2\). Thus, for any finite, non-empty subset \(F \in L\), \(\bigvee\) and \(\bigwedge\) exist.

Thus, every finite lattice bounded (has a greatest and least element) with

$$\top = \bigvee L \quad \bot = \bigwedge L$$

Finally, because finite lattices have \(\bot\) (\(\top\)), it exists \(\bigvee \emptyset\) (\(\bigwedge \emptyset\)):

$$\bot = \bigvee \emptyset \quad \top = \bigwedge \emptyset$$

Hence, finite lattices are complete.
Fixpoint by Iteration (Kleene)

Let $L$ be a complete lattice, $f : L \rightarrow L$ a monotone function, and $\alpha := \bigcup_{i \geq 0} f^i(\bot)$.

1. If $\alpha$ is a fixpoint, it is the least fixpoint.
2. If $f$ is continuous, $\alpha$ is a fixpoint.
Fixpoint by Iteration (Kleene)

Let $L$ be a complete lattice, $f : L \rightarrow L$ a monotone function, and $\alpha := \bigsqcup_{i \geq 0} f^i(\bot)$.

1. If $\alpha$ is a fixpoint, it is the least fixpoint.
2. If $f$ is continuous, $\alpha$ is a fixpoint.

Proof: First, $\alpha$ exists because $L$ is a lattice.

1. Assume $\beta = f(\beta)$ is a fixpoint of $f$. By definition, $\bot \leq \beta$ and because $f$ is monotone, for all $i$: $f^i(\bot) \leq f^i(\beta) = \beta$. Hence, $\beta$ is an upper bound on $M = \{\bot, f(\bot), \ldots\}$. Because $\alpha$ is the least upper bound of $M$, we have $\alpha \leq \beta$. Hence, if $\alpha$ is a fixpoint, it is the least.

2. $f(\alpha) = f(\bigsqcup_{i \geq 0} f^i(\bot)) = \bigsqcup_{i \geq 0} f(f^i(\bot))$ if continuous
   $= \bigsqcup_{i \geq 1} f^i(\bot)$
   $= \bigsqcup_{i \geq 0} f^i(\bot)$ because $\forall i. \bot \leq f^i(\bot)$
   $= \alpha$

Remark: The theorem also holds for complete partial orders in which only every ascending chain must have a least upper bound.
Fixpoints in Complete Lattices

\[ f(x) \leq x \]

\[ x = f(x) \]

\[ x \leq f(x) \]